Short Papers

Characteristic Impedance of a Coaxial System Consisting of Circular and Noncircular Conductors

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Abstract —**A family of transmission lines is based on a circular conductor and a noncircular conductor. Two new types of equivalent eccentric coaxial lines, which give smooth transition between extremes of a small wire and a wire near contact, are presented. The results obtained are very simple analytical expressions which will be useful for fast computation or for the CAD of coaxial components. The accuracy of the expressions is confirmed by comparison with accurate numerical data.**

Nomenclature

Z_0 Characteristic impedance of a transmission line.
r Radius of a circle circumscribed about an inner conductor.
R Radius of a circle inscribed in an outer conductor.
r_e Effective radius of an inner conductor.
R_e Effective radius of an outer conductor.
$l_e = (R_e - r_e)/(R - r)$ = normalized effective distance between inner and outer conductors.

The medium is taken to be free space.

I. Introduction

The determination of the characteristic impedance of a coaxial system consisting of a circular conductor and a noncircular conductor has been the subject of numerous treatments appearing during the past 40 years [1]–[15]. When the geometrical parameters of a coaxial transmission line are specified, we may calculate its impedance (or capacitance) using three approaches: 1) conformal transformation; 2) numerical techniques; and 3) graphically approximate methods, which identify an equivalent coaxial transmission line whose impedance is well known and is expected to be similar to that of the one under investigation. The third method has been used extensively to produce an equivalent circular coaxial line at small ratios of inner and outer conductors [2], [3]. However, this approach does not take into account the interaction of inner and outer conductors; thus the equivalent circular coaxial line is not a satisfactory approximation. Some improvements were made by using conformal transformation techniques and taking the arithmetic or geometric means of the upper and lower bounds to the size (or the upper and the lower bounds on the characteristic impedance) [7]. However, this requires rather tedious calculations to determine the bounds and only applies to some particular configurations. Recently an equivalent eccentric coaxial line was proposed and an elementary formula was presented for the determination of the characteristic impedance of a coaxial line consisting of a noncircular outer conductor and a circular inner conductor [15]. However, the formula has the maximum absolute error of the characteristic impedance for moderate ratios of inner and outer conductors, which is not desirable for practical use.

In this paper, we further develop the approximate graphical method and present two new types of equivalent eccentric coaxial lines, whose eccentricities vary with the ratio of inner and outer conductors, for a coaxial system consisting of circular and noncircular conductors. The new equivalent lines give smooth transition between extremes of a small wire and a wire near contact. The results obtained are very simple analytical expressions which will be useful for fast computation of the characteristic impedance or for the CAD of coaxial components. In comparison with the existing data, all our results are as accurate as the data available in the literature. Some results are believed to be better than those reported in the literature or are presented for the first time.

II. Theory and Method

A. The Transformation of Simple Connected Regions by an Infinite Series

The interior of a unit circle in the ζ plane can be conformally mapped into the interior of a simple connected region in the w plane by an infinite series of the form [16]

$$W = \sum_{n=0}^{\infty} \alpha_n \zeta^{1+n} \tag{1}$$

where

$$\alpha_n = a_n + ib_n. \tag{2}$$

If $|\zeta| \ll 1$, the first term in (1) is predominant, and a circle with radius $r \ll 1$ in the w plane will map into an approximate circle in the $|\zeta|$ plane with the radius

$$r' = \frac{r}{|\alpha_0|}. \tag{3}$$

When the region in the w plane has p axes of symmetry, b_n in (2) is zero and (1) can be rewritten as

$$W = \sum_{n=0}^{\infty} a_n \zeta^{1+pn}. \tag{4}$$

The exterior of a unit circle in the $|\zeta|$ plane can be conformally mapped onto the exterior of a simply connected region in the w plane by an infinite series of the form [16]

$$W = \sum_{n=0}^{\infty} \beta_n \zeta^{1-n} \tag{5}$$

where

$$\beta_n = c_n + id_n. \tag{6}$$

If $|\zeta| \gg 1$, the first term in (5) is predominant, and a circle with radius $R \gg 1$ in the w plane will map into an approximate circle in the ζ plane with the radius

$$R' = \frac{R}{|\beta_0|}. \tag{7}$$

When the region in the w plane has p axes of symmetry, d_n in

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(6) is zero and (5) can be rewritten as

$$W = \sum_{n=0}^{\infty} c_n \zeta^{1-pn}. \tag{8}$$

The coefficients α_n and β_n can be systematically determined by numerical methods (successive approximations, Melentiev's method, etc. [16]). The coefficient α_0 or β_0 can also be obtained by means of the closed analytic function which conformally maps a unit circle in the ζ plane into a simply connected region in the w plane.

B. Basic Properties of the Effective Radius of a Coaxial Conductor

The noncircular conductor of a coaxial line can be replaced by a concentric circular one with an effective radius. From the viewpoint that the total capacitance of a coaxial transmission line is composed of the parallel connection of the capacitance of every segment of the boundary, we have the following interesting conclusion about the properties of the effective radius of a coaxial conductor: The normalized effective distance l_e of a coaxial system consisting of circular and noncircular conductors reduces monotonically as the ratio r/R increases; that is, the effective radius R_e of the noncircular outer conductor decreases and the effective radius r_e of the noncircular inner conductor increases as the ratio r/R increases.

The proof of the above conclusion is obvious from physical considerations. As the ratio r/R increases, the contribution of the capacitances of the nearer boundaries to the total capacitance of a coaxial line rises and then the normalized effective distance l_e decreases. The above conclusion has been confirmed by checking the eccentrical coaxial line.

Moreover, if one of two conductors of a coaxial transmission line is chosen as the reference boundary, the normalized effective distance l_e between the reference boundary and the equivalent conformal boundary of the other conductor is a monotonically decreasing function of r/R.

C. New Type of Equivalent Coaxial Lines

When the ratio $r/R \to 0$, the effective radius of a noncircular outer conductor is given from (7) with close approximation by

$$R_{e0} = |\alpha_0| R \tag{9}$$

where $|\alpha_0|$ is usually referred to as a shield factor, and the effective radius of a noncircular inner conductor is given from (8) with close approximation by

$$r_{eo} = |\beta_0| r. \tag{10}$$

Generally, it is difficult to solve for the effective radius R_e or r_e exactly as the ratio r/R increases for an arbitrarily irregular coaxial conductor. Considering the basic properties of the effective radius given above, we propose to use two eccentric coaxial lines whose eccentricities vary with the ratios of inner and outer conductors as the equivalent coaxial line. One is for a coaxial system consisting of a noncircular outer conductor and a circular inner conductor, and the other is for a coaxial system consisting of a circular outer conductor and a noncircular inner conductor. At the extreme of a small wire ($r/R \to 0$), the eccentricities of the equivalent lines are zero and the equivalent lines become circular coaxial lines, with the effective radius R_{e0} in (9) for a noncircular outer conductor and r_{e0} in (10) for a noncircular inner conductor.

At the extreme of a large wire near contact ($r/R \to 1$), the inner and outer conductors of the equivalent lines should be near contact. For the case $r/R = 1$, the eccentricity of the equivalent lines reaches the maximum value, and can be determined easily from the geometrical configuration of a eccentric coaxial line. In particular, the maximum eccentricities of the equivalent lines are

$$E_{\max} = \left(1 - \frac{1}{|\alpha_0|}\right) \tag{11}$$

for a coaxial system consisting of a noncircular outer conductor and a circular inner conductor, and

$$E_{\max} = 1 - |\beta_0| \tag{12}$$

for a coaxial system consisting of a circular outer conductor and a noncircular inner conductor. It will be shown in Section III that the characteristic impedances obtained from the equivalent line with the maximum eccentricity in (11) are very close to Wheeler's limiting results for polygons [4].

Between the extremes of a small wire and one near contact, we may choose the eccentricity of an equivalent eccentric line such that its effective radius is very close to that of the one under investigation. We use $|\alpha_0|$ and $|\beta_0|$ to indicate the departure of a noncircular conductor from a circular one. Thus, the eccentricities of the equivalent line should be a function of $|\alpha_0|$, $|\beta_0|$, and r/R.

Although it is difficult to solve for the effective radius of a coaxial conductor of any irregular cross section at any ratio r/R, there are some combinations of circular and noncircular conductors whose effective radii can be evaluated exactly. These can be used to determine the function of the eccentricity of the equivalent line with the variables $|\alpha_0|$, $|\beta_0|$, and r/R by means of optimization techniques, such as optimum seeking methods.

Once such a function is determined, the characteristic impedance at any ratio r/R of a coaxial system consisting of circular and noncircular conductors can be easily calculated by the formula for the determination of the characteristic impedance of the eccentric coaxial line. In the following, two such functions will be given. One is for a noncircular outer conductor and the other is for a noncircular inner conductor.

III. Noncircular Outer Conductor

By the optimum seeking method, the eccentricity of the equivalent eccentric line for a coaxial system consisting of a noncircular outer conductor and a circular inner conductor is chosen and is written in terms of the variables $|\alpha_0|$ and r/R as

$$E_1(r/R) = \left(1 - \frac{1}{|\alpha_0|}\right)(r/R)^{F_1(r/R)} \tag{13}$$

where

$$F_1(r/R) = \frac{2}{|\alpha_0|}\left[1 - \left(\frac{r}{R}\right)^{10/|\alpha_0|}\right]. \tag{14}$$

Then, the formula for the determination of the characteristic impedance of this kind of coaxial transmission line is given by

$$Z_0 = 59.952 \ln\left[G + \sqrt{(G^2 - 1)}\right] \tag{15}$$

where

$$G = \frac{1}{2}\left\{\frac{2r}{|\alpha_0|R} + \frac{|\alpha_0|R}{r}\left[1 - E_1(r/R)\right]\left[1 - \frac{(1 - E_1(r/R))}{2}\right]\right\}. \tag{16}$$

To show the validity of the formula given above, we consider some typical examples. Fig. 1 shows a family of outer conductor cross sections that are regular polygons. The variable a_0 for

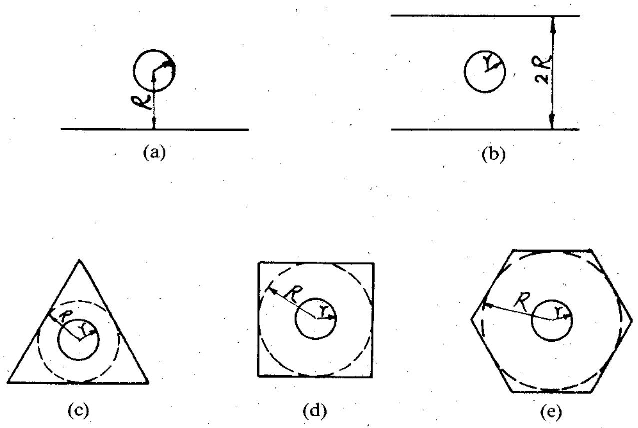

Fig. 1. Outer conductor of regular-polygon cross section for (a) $N=1$, (b) $N=2$, (c) $N=3$, (d) $N=4$, and (e) $N=6$.

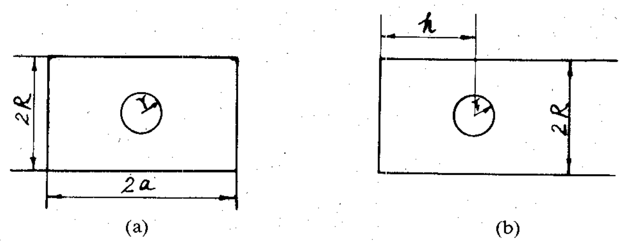

Fig. 2. Outer conductor of nonregular-polygon cross section.

TABLE I
CHARACTERISTIC IMPEDANCE FOR N-REGULAR-POLYGON OUTER CONDUCTOR

N	1		2		6	
r/R	present work	Gunston [1, p.16]	present work	Wheeler [4]	present work	Seshagiri [1, p.89]
0.05	221.12	221.12	194.08	194.08	181.82	181.64
0.1	179.45	179.45	152.52	152.52	140.26	140.01
0.3	112.34	112.34	86.59	86.62	74.40	74.04
0.5	78.95	78.95	55.66	55.72	43.77	43.43
0.7	53.69	53.69	34.51	34.54	23.56	23.34
0.9	28.01	28.01	16.03	16.07	8.30	8.25
0.95	19.37	19.37	10.64	10.67	4.85	4.83

N	3			4		
r/R	present work	Seshadri [11]	Epele [13]	present work	Seshadri [11]	Riblet [6]
0.05	187.04	187.32		184.14	184.42	
0.1	145.48	145.70	145.50	142.59	142.50	
0.3	79.61	79.74	79.63	76.72	76.84	
0.5	48.91	49.03	48.98	46.07	46.16	46.09
0.7	28.43	28.57	28.53	25.77	25.89	25.85
0.9	11.99	12.06	12.10	10.06	10.15	10.13
0.95	7.70			6.24		6.25

TABLE II

N	1	2	3 triangle	3 trough	4	6	8
a_0	2	1.2732	1.1321	1.1678	1.0787	1.0376	1.0220
1/N	1	0.5	0.333	0.333	0.25	0.167	0.125
$\sqrt{1-\frac{1}{a_0}}$	0.707	0.463	0.342	0.379	0.27	0.190	0.147

polygons can be formulated in terms of Gamma function [3], [17]:

$$a_0 = \frac{\Gamma\left(1+\frac{2}{n}\right)}{\Gamma^2\left(1+\frac{1}{n}\right)}. \tag{17}$$

Substituting (17) into (13)–(15) yields simple analytical expressions for the characteristic impedance of a coaxial system consisting of N regular outer conductors and a circular inner conductor. Table I shows a comparison between the characteristic impedance reported in the literature and that obtained using these simple expressions.

It is shown in this table that the agreement is excellent in all cases. In particular, the maximum deviation is less than 0.2 percent in the range $r/R < 0.5$ and less than 0.7 percent in the range $r/R < 0.95$. At the extreme of large wires near contact, we find, by taking the limit of (15), that

$$Z_0 \doteq 59.952\sqrt{2\left(1-\frac{1}{|\alpha_0|}\right)\left(\frac{R}{r}-1\right)}. \tag{18}$$

TABLE III
CHARACTERISTIC IMPEDANCE FOR RECTANGULAR OUTER CONDUCTOR

a/R	1.5			2			2.5		
r/R	present work	Lin [1, p.66]	Pan [14]	present work	Lin [1, p.66]	Pan [14]	present work	Lin [1, p.66]	Pan [14]
0.05	191.95	191.95	191.95	193.64	193.64	193.64	193.99	193.99	193.99
0.1	150.39	150.39	150.46	152.08	152.08	152.13	152.43	152.43	152.44
0.3	84.48	84.50	84.65	86.15	86.19	86.15	86.51	86.55	86.32
0.5	53.64	53.70	53.81	55.24	55.39	55.07	55.58	55.75	55.09
0.7	32.72	32.82	32.94	34.14	34.48	33.86	34.43	34.82	33.78
0.9	14.88	15.14	15.35	15.79	16.44	15.73	15.98	16.71	15.66
0.95	9.81	10.10	10.27	10.47	11.11	10.46	10.61	11.32	10.40
0.99	4.12	4.31	4.36	4.42	4.80	4.39	4.48	4.90	4.37

TABLE IV
CHARACTERISTIC IMPEDANCE FOR TROUGH OUTER CONDUCTOR

h/R	1/4		1/2			3/4	
r/R	present work	Chisholm [1, p.74]	present work	Wheeler [4]	Chisholm [1, p.74]	present work	Chisholm [1, p.74]
0.05	210.34		188.90	188.90	188.89	193.01	193.01
0.1	168.75	168.72	147.34	147.34	147.34	151.45	151.45
0.3	102.37	102.26	81.47	81.39	81.36	85.54	85.55
0.5	70.23	70.27	50.71	50.52	50.46	54.65	54.69
0.7	46.71	47.41	30.09	29.69	29.67	33.62	33.69
0.9	23.60	26.49	13.13	12.56	13.24	15.46	16.46
0.94	17.78	20.36	9.52	8.95	10.19	11.36	12.72
0.98	9.97	16.77	5.11	4.61		6.19	

Equation (18) may be compared to Wheeler's limiting result for polygons [4]:

$$Z_0 \doteq \frac{59.952}{N}\sqrt{2\left(\frac{R}{r}-1\right)}. \tag{19}$$

To the extent that $1/N = \sqrt{1-1/|\alpha_0|}$, (18) and (19) give the same results and the comparison is shown in Table II. As can be seen, the results are very close in most cases. Fig. 2 shows cross sections which are nonregular polygons.

The variable a_0 for the rectangle is given by [2], [3]

$$a_0 = \frac{4}{\pi}\exp\left(\frac{-4}{\exp\left(\frac{\pi a}{R}\right)+1}\right) \tag{20}$$

and a_0 for the trough is given by [1], [2]

$$a_0 = \frac{4}{\pi}\tanh\left(\frac{\pi h}{R}\right). \tag{21}$$

The results obtained by substituting (20) and (21) into (13)–(15) are compared with earlier published results in Tables III and IV. These tables show that the present values are in excellent agreement with those of Lin and Chisholm for $r/R < 0.7$. At the extreme of a large wire near contact, we find that the present results are closer to Wheeler's limiting results for polygons of (19) than those of Lin and Chisholm. So our present results are thought to be better than those quoted by Gunston [1].

Fig. 3 shows a elliptical cross section, which is a family of simple closed curves depending on one parameter λ. The equation of the ellipse can be expanded in a power series in the parameter λ and the mapping function can be found by successive approximations.

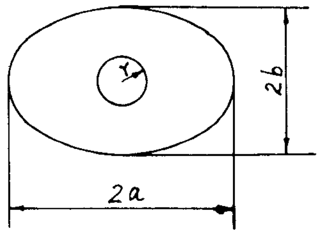

Fig. 3. Outer conductor of elliptical cross section.

TABLE V
CHARACTERISTIC IMPEDANCE FOR ELLIPTICAL OUTER CONDUCTOR

r/b \ a/b	1.5	2	3	5
0.05	188.27	189.92	192.14	193.30
0.1	146.71	148.36	150.58	151.74
0.3	80.83	82.47	84.67	85.82
0.5	50.10	51.70	53.82	54.93
0.7	29.53	30.98	32.88	33.86
0.9	12.75	13.73	14.99	15.61

The parameter a_0 for an ellipse is given by

$$a_0 = \sqrt{1+\lambda}\left(1 - \frac{\lambda^2}{8} + \frac{3}{128}\lambda^4\right) \tag{22}$$

where the semi-axes a and b are $(1-\lambda)^{-1/2}$ and $(1+\lambda)^{-1/2}$, respectively. The accuracy of (22) is close to the fifth power of λ. For the extreme case ($\lambda = 1$), the ellipse becomes parallel planes and the relative error of (22) is only about 0.2 percent.

When (22) is substituted into (13)–(15), we obtain the formula for the determination of the characteristic impedance. The results for various values of the ratios a/b and r/R are summarized in Table V. Comparable results are not available in the literature.

IV. NONCIRCULAR INNER CONDUCTOR

By the optimum seeking method, the eccentricity of the equivalent eccentric line for a coaxial system consisting of a circular outer conductor and a noncircular inner conductor is chosen and written in terms of the variables $|\beta_0|$ and r/R as

$$E_2(r/R) = (1-|\beta_0|)\left(0.66 + \frac{|\beta_0|}{10}\right)^{g(r/R)}\left(\frac{r}{R}\right)^{F_2(r/R)} \tag{23}$$

where

$$g(r/R) = 1 - \left(\frac{r}{R}\right)^{80} \tag{24a}$$

$$F_2(r/R) = \left(3\sqrt{|\beta_0|}\right)^{[1-(r/R)^{15}]}. \tag{24b}$$

Then, the formula for the determination of the characteristic impedance for this kind of coaxial transmission line is given by

$$Z_0 = 59.952 \ln\left[G + \sqrt{(G^2-1)}\right], \tag{25}$$

where

$$G = \frac{1}{2}\left\{\frac{2|\beta_0|r}{R} + \frac{R}{|\beta_0|r}\left[1 - E_2(r/R)\right]\left[1 - \frac{(1-E_2(r/R))}{2}\right]\right\}. \tag{26}$$

Fig. 4 shows a family of inner conductor cross sections that are regular polygons.

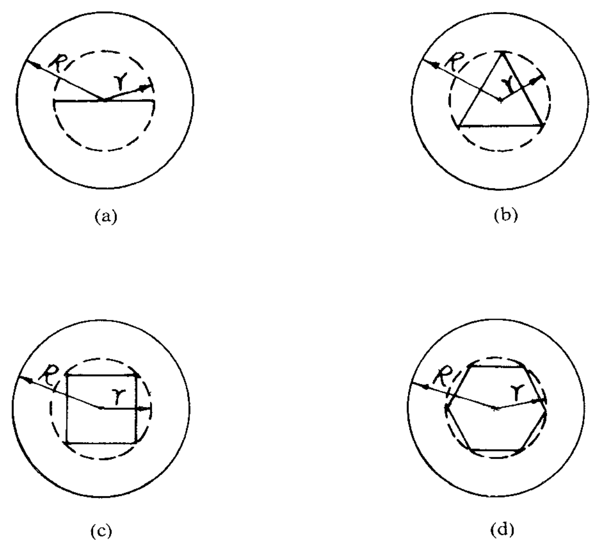

Fig. 4. Inner conductor of regular-polygon cross section for (a) $N = 2$, (b) $N = 3$, (c) $N = 4$, and (d) $N = 6$

TABLE VI
CHARACTERISTIC IMPEDANCE FOR N-REGULAR-POLYGON INNER CONDUCTOR

N	3			4			
r/R	present work	Sheshadri [12]	Fan [14]	present work	Sheshadri [12]	Lin [7]	Fan [14]
0.05	156.87	155.20	156.87	169.66	169.57	169.67	169.66
0.1	115.31	113.56	115.19	128.11	127.95	128.11	128.12
0.3	49.24	47.40	48.31	[illegible]	61.99	[illegible]	[illegible]
0.4	31.01	28.86	29.85	44.94	44.70	44.97	[illegible]
0.5				31.40	31.20	31.49	31.38
0.6				19.87	[illegible]	20.05	19.95
0.65				14.24	14.08		
0.7				[illegible]		[illegible]	[illegible]

N	2			6		
r/R	present work	Gunston [1, p.80]	Oberhettinger [17]	present work	Sheshadri [12]	Fan [14]
0.05	221.16	221.15		175.95		175.95
0.1	179.60	179.60		134.39		134.41
0.3	113.69	113.67		68.54	68.59	68.59
0.6	71.22	71.76		26.93	26.96	27.08
0.7	60.99	60.95		17.58		17.79
0.9	40.80	40.91	40.13			
0.94	36.21	36.00	35.25			
0.99	25.03		24.71			

For polygons c_0 can be formulated in terms of Gamma function [3], [17]:

$$c_0 = \frac{n\Gamma^2\left(1+\frac{1}{n}\right)\sin\frac{\pi}{n}}{\pi\Gamma\left(1+\frac{2}{n}\right)}. \tag{27}$$

Substituting (27) into (23)–(25) yields simple analytical expressions for the characteristic impedance of a coaxial system consisting of a circular outer conductor and an N-regular polygon inner conductor. Table VI shows a comparison between the characteristic impedance reported in the literature and that obtained using these simple expressions. As can be seen from the table, the agreement is excellent in most cases. In particular, the maximum deviation is less than 0.3 percent in the range $r/R < 0.9$ for $N = 2$, 4, and 6. For $N = 3$, our results are almost 1.5 Ω greater than those of Sheshadri for different r/R ratios [12]. By small wire theory [2], the results reported in this paper are found to be more accurate than those reported by Sheshadri.

V. CONCLUSIONS

A new type of equivalent eccentric coaxial line has been presented for general coaxial systems. The elementary formulas for characteristic impedance are given in a form that can be implemented easily on a pocket calculator. In terms of accuracy, the described formulas represent a considerable improvement on the foundation of coaxial component design and are fully compatible with the needs of modern computer-aided microwave coaxial circuit design.

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Mode Stability of Radiation-Coupled Interinjection-Locked Oscillators for Integrated Phased Arrays

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***Abstract* — An array of coupled oscillators can synthesize the microwave phase relationships needed for phased arrays by means of a technique known as interinjection locking. The mode required must be stable, and a general approach for evaluating mode stability and predicting frequency and phase relationships is applied to an experimental two-element 10 GHz array. Radiation coupling between the two oscillators leads to coherent operation, and the simple theory developed successfully predicts the system's behavior over a wide range of interoscillator distances.**

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I. Introduction

Recently a novel method of phase generation and control for phased arrays has been developed. The technique, described as interinjection locking [1], [2] or parasitic injection locking [3], consists in driving each element in a phased antenna array with its own directly coupled oscillator. Suitable coupling between the antenna elements causes the system as a whole to run coherently, synthesizing the properly phased drive for each element. Rather than requiring a phase shifter for each element, the system can be steered by a few externally controlled injection inputs at strategic points. The difficulties and losses encountered when a phase-shifting circuit must be provided for each element could be greatly reduced by applying this technique to microwave, and especially millimeter-wave, integrated circuits in which phase shifter losses rise rapidly with increasing frequency.

A system of N nonlinear oscillators can in principle operate in any one of N single-frequency modes, and even more if multiple-frequency operation is considered. Typically only one of these modes meets the phased array requirements, so some means must be established for evaluating mode stability in systems of coupled oscillators. In this paper we analyze two oscillators coupled solely by means of the free-space interaction between their respective antenna elements. The oscillators are modeled as energy-storing $L-C$ tank circuits in parallel with voltage-dependent negative conductances. A simplified far-field slot antenna model is used to derive the mutual admittance of the two antennas. Even-odd mode analysis yields the normal modes of the system, and a theorem from averaged potential theory is used to determine which mode is stable. Two microstrip Gunn diode oscillators were built to verify the essential features of the model. Oscillator frequencies, relative phases, and radiation patterns were measured as functions of the interantenna distance, and the periodic alternation of modes with distance predicted by theory was confirmed quite well. Although the small system studied is of limited practical use, it has many features in common with larger practical interinjection-locked systems.

II. Theory

Most studies of multiple-device oscillators, such as Kurokawa's [4], assume that the primary energy storage mechanism is a resonant-structure mode common to all the devices, with relatively little energy stored within each device's associated circuitry. In contrast to this, the interinjection-locking approach begins with self-sufficient oscillators capable of independent operation, but susceptible to injection locking with a signal applied to their outputs. In Fig. 1 two such oscillators are modeled as parallel equivalent circuits consisting of $L-C$ tanks and voltage-dependent negative conductances $-G_D(v)$. For most purposes a simple cubic function $-G_D(v) = -g_1 v + g_3 v^3$ suffices to model devices in systems with relatively small levels of injection power [5]. Circuit losses are modeled by conductances G_L, and in the absence of external loads ($I_1 = I_2 = 0$), each circuit will achieve a steady-state oscillation at a frequency $\omega_0 = 1/\sqrt{LC}$ and amplitude V_0 such that $-G_D(V_0) + G_L = 0$. In the discussion that follows, the oscillators are assumed to have identical characteristics.